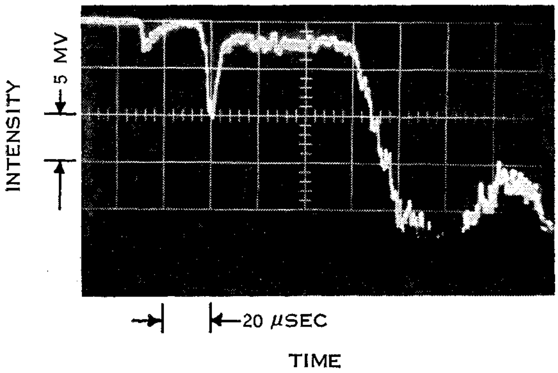

Fig. 2 Radiation profile with $Cr(CO)_6$ in buffer; initial test pressure 0.2 mm Hg, M_s = 13.0.

air. The equilibrium radiation that follows the chromium overshoot is greater, however, and the final abrupt rise is somewhat damped.

There appears to be no question that the new spike is caused by chromium. Similar tests, using carbon monoxide in the buffer, were run and produced no intermediate overshoot. If the chromium spike is considered the leading edge of the contact surface, then the uniform flow time is considerably shorter than previously believed. The rapid quenching of the chromium radiation is attributed to the fact that the argon in the buffer inefficiently transfers energy to the chromium by collision, and therefore the chromium temperature is effectively 400° to 500°C cooler than would be expected with a diatomic buffer gas.[3]

Roshko[4] and others[5, 6] have discussed the problem of flow duration in low-pressure shock tubes. By analyzing the effects of the laminar boundary-layer buildup behind the shock wave, Roshko has developed a theory that relates the various parameters affecting the uniform flow time such as initial test pressure, shock Mach number, and tube diameter. We have used this theory to calculate the uniform flow time for our tube using real gas properties for the various experimental conditions.

Figures 3 and 4 plot flow duration against initial pressure in the test section at Mach 12 and 15, respectively, and include a curve representing the calculated maximum values. The circles are experimental times from initial air radiation overshoot to chromium peak, and the squares are times from initial overshoot to final rise in radiation intensity.

In summary, the buffer gas, as indicated by the chromium carbonyl experiments, arrives substantially earlier than previously assumed from other indications of arrival of the contact region. Computed flow duration values (an upper limit, according to Roshko's treatment) correlated well with the time of appearance of the chromium spike for the higher test pressures. At very low pressures, however, flow times appear to be longer (Figs. 3 and 4) than predictions based on Roshko, and may indicate the onset of a slip condition at the

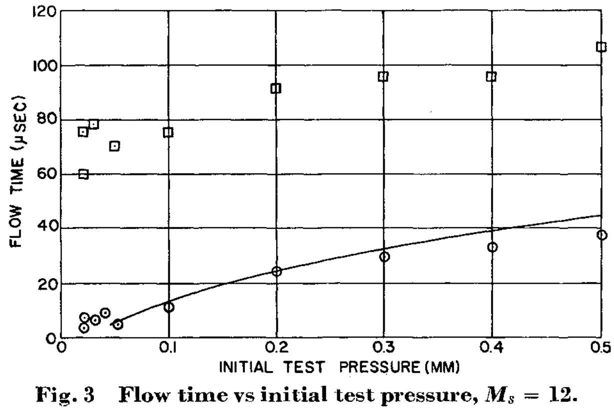

Fig. 3 Flow time vs initial test pressure, M_s = 12.

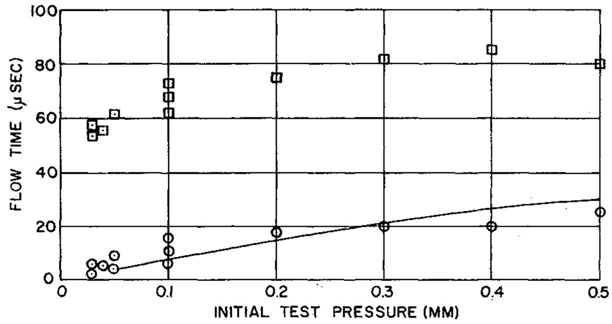

Fig. 4 Flow time vs initial test pressure, M_s = 15.

walls. Preliminary data at even lower pressures appear to substantiate this difference. This work is continuing.

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Turbulent Boundary Layers with Transpiration

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A UNIVERSAL equation is presented for the mean velocity in the outer region of turbulent boundary layers with suction or injection through a porous wall. The equation includes $F(y/\delta)$, the universal function of y/δ which occurs in the velocity defect equation for zero pressure gradient and zero blowing velocity, but the equation contains no additional empirical constants. Many authors have attempted, with limited success, to generalize the velocity defect equation for zero pressure gradient to include the cases of suction and injection. Mickley and Smith,[1] for instance, propose an equation for the outer region of turbulent boundary layers with injection, but their equation possibly needs modifying when there is suction. Mickley and Smith's equation is

$$(u_1 - u)/u_\tau^* = F(y/\delta) \qquad (1)$$

where $F(y/\delta)$ is the same function as that in the velocity defect law for zero pressure gradient and zero blowing velocity, and $u_\tau^*[=(\tau/\rho)^{1/2}]$ is based on the maximum total shear stress. When there is zero blowing velocity, the maximum shear stress occurs at the wall, and Eq. (1) is then the accepted velocity defect law for zero pressure gradient and zero blowing velocity. When there is suction, the maximum shear stress again occurs

Received April 15, 1964. The author wishes to thank G. M. Lilley of The College of Aeronautics and A. D. Young of Queen Mary College, The University of London, for the many helpful discussions during the course of this work.

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at the wall; however, Black and Sarnecki[2] were unsuccessful when they tried to use this equation with suction.

The equation for the inner region of a turbulent boundary layer with transpiration, which may be derived by a mixing length or equilibrium theory,[3] is

$$\frac{2u_\tau}{v_w}\left\{\left(1+\frac{v_w u}{u_\tau^2}\right)^{1/2}-1\right\}=\frac{1}{K}\log e\,\frac{yu_\tau}{\nu}+B \qquad (2)$$

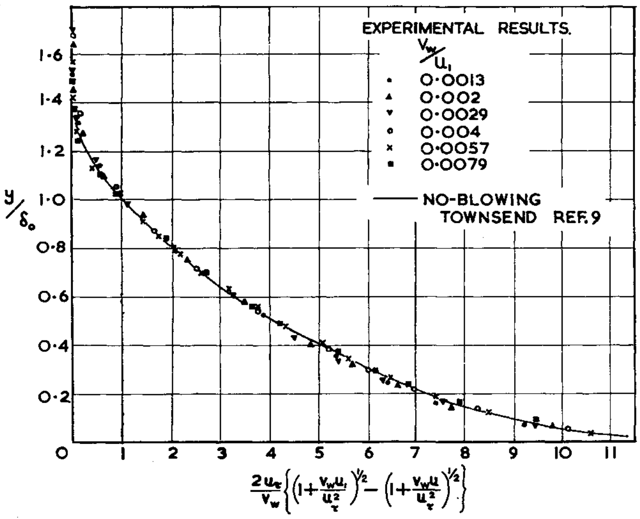

Fig. 1 The curve for the outer region [δ_0 is the value of y at which $F(y/\delta) = 1$].

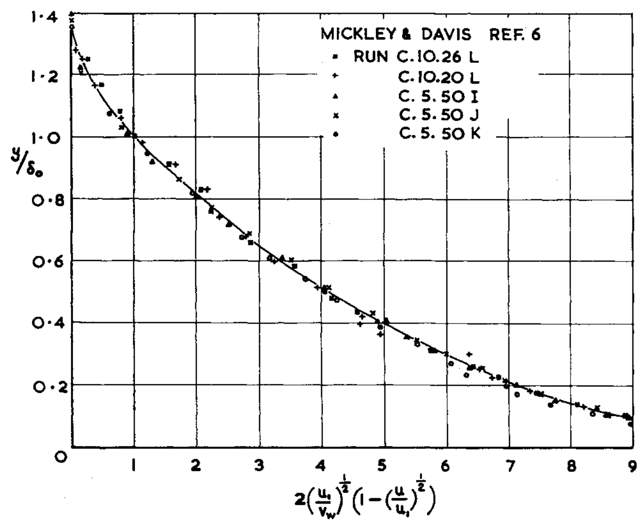

Fig. 2 The curve for the outer region.

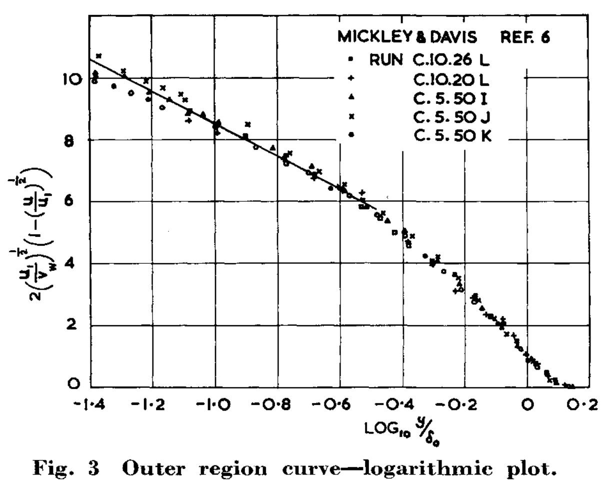

Fig. 3 Outer region curve—logarithmic plot.

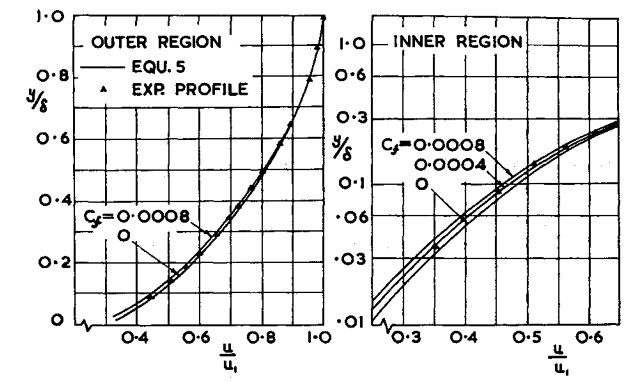

Fig. 4 Velocity profiles with injection; $(v_w/u_1) = 0.0057$.

where B is, in general, a function of v_w and u_τ. v_w is the blowing velocity normal to the porous surface, and $u_\tau[=(\tau_w/\rho)^{1/2} = u_1(c_f/2)^{1/2}]$ is the friction velocity at the wall.

By methods analogous to those of von Kármán or Millikan,[4] Eq. (2) is used to derive the equation for the outer region with transpiration,[5]

$$\frac{2u_\tau}{v_w}\left\{\left(1+\frac{v_w u_1}{u_\tau^2}\right)^{1/2}-\left(1+\frac{v_w u}{u_\tau^2}\right)^{1/2}\right\}=F\left(\frac{y}{\delta}\right) \qquad (3)$$

The author's experimental results agree very well with Eq. (3). A few of the experimental results are shown in Fig. 1.

When the blowing velocity is large, $(v_w/u_1) > 0.005$ or $(v_w u/u_\tau^2) \gg 1$, the outer region is governed primarily by the blowing velocity ratio v_w/u_1, and Eq. (3) reduces to

$$2\left(\frac{u_1}{v_w}\right)^{1/2}\left\{1-\left(\frac{u}{u_1}\right)^{1/2}\right\}=F\left(\frac{y}{\delta}\right) \qquad (4)$$

In Figs. 2 and 3, a selection of the experimental results of Mickley and Davis[6] are given, and it is shown that they compare favorably with Eq. (4).

The velocity distribution can be obtained from the following rearranged form of (3):

$$\frac{u}{u_1}=1-F\left(\frac{y}{\delta}\right)\cdot\frac{u_\tau}{u_1}\left(1+\frac{v_w u_1}{u_\tau^2}\right)^{1/2}+\frac{v_w}{4u_1}\left\{F\left(\frac{y}{\delta}\right)\right\}^2 \qquad (5)$$

and may be evaluated for a series of values of c_f for any transpiration velocity. The velocity profiles so obtained agree very well with the author's experimental results. One velocity profile is shown as an example in Fig. 4. The skin friction obtained from momentum traverses was approximately 0.0003.

In Fig. 5, it is shown that Eq. (5) also compares favorably with a near asymptotic suction velocity profile as measured by Dutton.[7]

Attention is drawn to the fact that Eq. (3) is only applicable to the case of zero external pressure gradient. The outer region of a turbulent boundary layer is very dependent on the pressure gradient, and a slight pressure gradient will modify

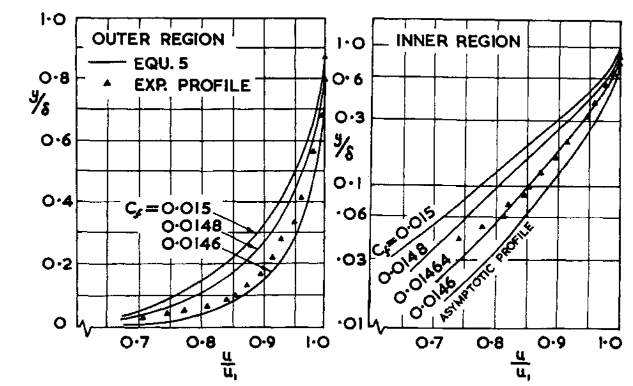

Fig. 5 Velocity profiles with suction; $(v_w/u_1) = -0.0073$.

the velocity profiles considerably. In conclusion, it might be noted that Eq. (3) is a special case of a more general law for the outer region of equilibrium turbulent boundary layers.[8]

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Exact Two-Body Error Sensitivity Coefficients

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Nomenclature

r	= radial distance to dynamical center
p	= semilatus rectum
n^*	= $(\mu/p^3)^{1/2}$ = modified mean motion parameter
v	= true anomaly
e	= eccentricity
τ	= $t - t_0$ = time since initial epoch
Δx	= inertial horizontal in-plane position deviation
Δy	= inertial out-of-plane position deviation
Δz	= inertial radial position deviation
$\Delta\dot{x}^*$	= inertial horizontal in-plane velocity deviation
$\Delta\dot{y}^*$	= inertial out-of-plane velocity deviation
$\Delta\dot{z}^*$	= inertial radial velocity deviation
μ	= gravitational constant of primary body
p/r	= $1 + ecv$ = nondimensional
cv, sv	= $\cos v$, $\sin v$
K_i	= constants of integration $i = (1,2, \ldots, 6)$
C_i	= arbitrary constants
W	= $sv(1 + ecv)$ = integrating factor
M_3	= $M_1\, cv + M_1'\, sv + 2(r/p)M_2$

Introduction

TABLE 1 of Ref. 1 presented a complete set of exact first-order error sensitivity coefficients for the general Keplerian reference orbit. This note presents an outline of its derivation. As noted in Ref. 1, there are two ways of approaching this solution. One method involves perturbing the two-body integrals with respect to the orbit constants relative to the final and initial epochs.[2] These perturbations are combined by elimination to yield the error sensitivity coefficients or the transition matrix. The other approach that is considered herein is to integrate directly the perturbation equations of motion. This latter approach is more pertinent to guidance theory.

Prior to publication of Ref. 1, in the recent literature there did not appear to be an exact, completely explicit solution to this problem by either method for a general Keplerian orbit. Danby[3] does give a complete solution by the former approach, but does not explicitly multiply out the 6×6 matrizants to yield the transition matrix. If he had multiplied the matrizants, the results of Table 1 of Ref. 1 would be obtained. Geyling[4] develops the general linear perturbation equations, but he also gives the solution on the basis of the former approach, stating, however, that the two approaches are equivalent. He does not give a complete solution to this problem and specializes to $v_0 = 0$ for in-plane perturbations.

The primary objective here is to tie the complete set of exact error coefficients (Table 1 of Ref. 1) to the linear perturbation equations of motion by outlining its derivation. Thereby, valuable insights can be gained on the nature of error propagation in a free-fall trajectory, on the structure of mechanization ("patched conic") of the navigation-guidance problem and on analytic error propagation for the nonhomogeneous solution required for the guidance problem. (Note: The transition matrix is the Green's matrix for the nonhomogeneous problem.) Further details on this derivation can be found in Refs. 2 and 5.

Derivation of Error Coefficients

The error coefficients are obtained upon solving the following linear perturbation equations of motion as an initial value problem:

$$\Delta\ddot{\mathbf{r}} - 2\omega\, \Delta\dot{\mathbf{r}} + (-\dot{\omega} + \omega^2)\, \Delta\mathbf{r} = -\frac{\mu}{r^3}\left[I - \frac{3\mathbf{r}\,\mathbf{r}^T}{r^2}\right]\Delta\mathbf{r} \tag{1}$$

where ω is the angular velocity matrix of the rotating frame. For a locally level in-plane out-of-plane coordinate system, one can write the components of $\Delta\mathbf{r}$

$$\Delta\mathbf{r} = \begin{bmatrix}\Delta x\\ \Delta y\\ \Delta z\end{bmatrix} = \begin{bmatrix}\text{inertial in-plane horizontal perturbation}\\ \text{inertial out-of-plane perturbation}\\ \text{inertial radial perturbation}\end{bmatrix} \tag{2}$$

and, if the reference trajectory is Keplerian, ω can be expressed as

$$\omega = \begin{bmatrix}0 & 0 & -\dot{v}\\ 0 & 0 & 0\\ \dot{v} & 0 & 0\end{bmatrix}; \quad \dot{v} = \frac{(\mu p)^{1/2}}{r^2} \tag{3}$$

The associated inertial velocity deviation in the local-level rectangular coordinates can be expressed as

$$\Delta\dot{\mathbf{r}}^* = \begin{bmatrix}\Delta\dot{x}^*\\ \Delta\dot{y}^*\\ \Delta\dot{z}^*\end{bmatrix} = \begin{bmatrix}\Delta\dot{x} + \dot{v}\,\Delta z\\ \Delta\dot{y}\\ \Delta\dot{z} - \dot{v}\,\Delta x\end{bmatrix} = \frac{(\mu p)^{1/2}}{r^2}\begin{bmatrix}\Delta x' + \Delta z\\ \Delta y'\\ \Delta z' - \Delta x\end{bmatrix} \tag{4}$$

where primes denote differentiation with respect to true anomaly.

Let the perturbation $\Delta\mathbf{r}$ be transformed

$$\Delta\mathbf{r} = \begin{bmatrix}\Delta x\\ \Delta y\\ \Delta z\end{bmatrix} = r\begin{bmatrix}L_x\\ L_y\\ L_z\end{bmatrix} = r\mathbf{L} \tag{5}$$

where $\mathbf{L}$ is an arbitrary vector.

Substituting Eq. (5) into Eq. (1), converting the time differentiation to true anomaly differentiation using two-body reference values, and taking due cognizance of the two-body

Received April 17, 1964.

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